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Function Approximation with Inertial Adaptive RBF Method

Keywords: *adaptive RBF, picture reconstruction*

1 Introduction

The radial basis function is one of the most popular methods on the field of scattered data approximation, and since some years it is a good tool to solve boundary value problems numericaly. The RBF method has many advantages against other wildspread numerical methods. It is simple, does not need any dicretization of the geometry. During the approximation it uses only center points and base function around them.

Recent years many functions were used as base function, like Gaussian, multiquadratic, spline or inverse multiquadratic functions. Many reseacher took publications about convergence ablilities, stability and robustness of these methods.

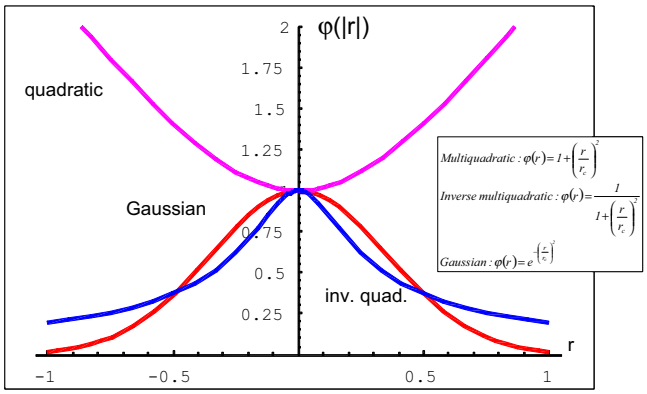


Figure 1: Types of RBF base functions

In many practical case, the goal is the approximation of the analitical solution with as few base function as it can be. This ceil can be achieved with using of multigrid RBF methods. In these methods the approximating function is built up in more step. Some of these methods use predefined centers on every stage of calculation, other methods use the residual of the last step to determine the place of new centers. The optimal choosing of new centers lead to a nonlinear optimization problem. There are many mathematical way to solve this problem, but most of them is qutie slow.

In our paper, we would like to show a new way, called Inertial Adaptive Radial Basis Function (IARBF) method, to determine the subsampling RBF centers, with a special technique. We start with equidistant centers, base functions and set of sampling points. From the centers and the radius of base functions the sampling points are grouped into subsets. After the first solution, made with minimization of different between the function and the approximation over the sampling points, the centroid of residuals is calculated in each subsets. The inertial matrix of residuals is also calculated. Knowing the matrix the first eigenvector is determined, which can split the subset into two subsubsets (children). The set of the children can be used as a new level of the radial basis. Coefficients of the new set are calculated two minimize the residual of the last level of radial basis approxiamation. This algorithm - calculation of coefficients, centroids, inetrial matrix, splitting the subsets - is repeated until the sum of residuals is small sufficient.

Our method is easy to implement and the mathematical problem of inertail matrix calculation and eigen value calculation is much simpler then numerical optimization. It works well and fast for lots of sampling point and base functions, too. By our experience, the IARBF is stable until the radius of base functions is larger then 2 times the distance between the sampling points. As it will be shown, the defined adaptive method is applicable on the field of function approximation. We show some example of function approximation in one and two dimension, on continous and discontinous functions. At last, the new method is applied in picture reconstruction.

2 Approximation with multigrid RBF functions

Let $f(\mathbf{x}) : \mathbf{x} \in R^n \longrightarrow R$ be the function to be approximated. Let $\Omega \subset R^n$ and $C := \{\mathbf{c}_i\}_{i=1}^N$ a set of centers in Ω. Let a set of $\{\varphi_i\}_{i=1..N}$ radial basis functions, defined as

$$\varphi_i : \mathbf{x} \in R^n \to R, \varphi_i(\mathbf{x}) = \varphi_i(\|\mathbf{x} - \mathbf{c}_i\|)$$

If $\|.\|$ is an Euclidean norm, then the φ_i is the function only the distance from the center c_i. To approximate the f function, the linear combination of the base functions is used:

$$\widetilde{f}(\mathbf{x}) := \sum_{i=1}^{N} d_i \varphi_i(\mathbf{x}) = \{\Phi\}^T \cdot \{d\} \approx f(x)$$

There are many possible way to define the base functions. Most widely used cases can be found here:

Thin plate spline:	$\varphi(r) = r^2 \ln 2$
Gaussian:	$\varphi(r) = e^{-\left(\frac{r}{a_i}\right)^2}$
Multi quadratic (MQ):	$\varphi(r) = \left(r^2 + a_i^2\right)^{\alpha}$
Inverse multi quadratic (IMQ):	$\varphi(r) = \left(r^2 + a_i^2\right)^{-\alpha}$
Spline:	$\varphi(r) = \left\{ \begin{array}{cc} \text{if } r < a_i & 1 - 2\left(\frac{r}{a_i}\right)^2 + \left(\frac{r}{a_i}\right)^4 \\ \text{else} & 0 \end{array} \right\}$

In our examples the MQ, Gaussian, and Spline function were used. Each of them has a parameter to adjust the shape of the base function which will be esential in our new method.

Let $X := \{x_i \in \Omega\}_{i=1..NP}$ a set of sampling points, with $NP > N$ condition. Let $F := \{f_i \in R\}_{i=1..NP}$ the sampled data. After definition of some $\|.,.\|_X$ norm on the X space, the determination os approximation function is equal with the minimization of the

$$J := \left\| f(\mathbf{x}) - \widetilde{f}(\mathbf{x}), f(\mathbf{x}) - \widetilde{f}(\mathbf{x}) \right\|_X$$

discrete functional.

In the case of multigrid RBF methods, there is $\left\{\varphi_i^j\right\}_{i=1..N_i}^{j=1..M,}$ a multilevel set of radial basis functions. Usually, $N_i < N_j$ when $i < j$. It means, set with higher index has more base functions and consequently higher accuarcy. In most techniqes, centers and base functions are defined in advance. The approximate solution is calculated on a multistep way. First, the most coarse of solution is defined with the first level of base functions. Then the residual of this function is calculated. The second level is used to approximate this residual. After this step, this algorithm can be continued till the last, most refined level of base functions.

$$\begin{aligned} r^j(\mathbf{x}) &= f(\mathbf{x}) - \widetilde{f}^{j-1}(\mathbf{x}) \\ \widetilde{f}^j(\mathbf{x}) &= \widetilde{f}^{j-1}(\mathbf{x}) + \sum_{i=1}^{N_j} d_i^j \cdot \varphi_i^j(\mathbf{x}) \end{aligned}$$

minimize the $J \quad : \quad = \left\| r^{j+1}(\mathbf{X}), r^{j+1}(\mathbf{X}) \right\|$ functional to determine d_i^j coefficients.

In most applications, the set of centers are spread uniformly over the Ω range. In our new method we use the information that contained in $\left\{R^j(X)\right\}$ sets to optimize the positions of set of centers C^{j+1} on $(j+1)^{th}$ level.

3 The itterative multigrid RBF method

3.1 Definition of itterative multigrid RBF method

Let us assume that, a C^0 set of centers and α^0 a set of nodal parameters are predefined. We define a global iterative procedure to generate a multigrid RBF method. During this procedure at each step of the itteration some of the bases will be split into two parts. So, we assume that each node can have two children a "son" and a "daughter" – or a left and a right one. At zeroth step, the zeroth residual have to be calculated at each sample point:

$$\left\{R^0\right\} := f\left(\mathbf{X}\right)$$

The system of linear equations

$$\begin{aligned} \sum_{i=1}^{N_0} \varphi_i^0\left(\mathbf{X}\right) \cdot d_i^0 &= R^0 \\ \left[M^0\right] \cdot \left\{d^0\right\} &= \left\{R^0\right\} \end{aligned}$$

is overdetermined and it can be solved by the mean of minimization of the error square:

$$\left[M^0\right]^T \cdot \left[M^0\right] \cdot \left\{d^0\right\} = \left[M^0\right]^T \cdot \left\{R^0\right\}$$

After solution the residual at the sample points can be calculated:

$$\left\{R^1\right\} := \left\{R^0\right\} - \sum_{i=1}^{N_0} \varphi_i^0\left(\mathbf{X}\right) \cdot d_i^0$$

Now, let us define the X_k^0 subset of the sampling points around the c_k^0 center points:

$$X_i^0 := \left\{\mathbf{x}_k \in \Omega, \varphi_i^0\left(\mathbf{x}_k\right) \neq 0\right\}$$

With these subsets, the residual of nodes – the *weight* of the i^{th} node – can be defined. Here we define it onto the j^{th} step of the itteration:

Definition 1 *Let $\left\{C^j\right\}$ is the set of centers on level the j^{th}. Let X_i^j the set of sample points "near" to c_i^j center. Let the weight of the c_i^j center is defined by*

$$w_i^j = \sum_{\mathbf{x}_k \in X_i^j} \left\| R^{j+1}\left(\mathbf{x}_k\right) \right\|$$

These weight are some measure of the quality of the base functions. Using these weights the nodes can be ordered. To refine the approxiamte solution some of the nodes will have two children. Each node can take children only one time. Each child will have his/her own center, parameter and coefficient.

Centers and parameters must be determined at the stage of borning, and the coefficients can be calculated at nest itteration step.

During the step of child producing, the by w_i^j weights of the nodes a criteria can be enforced to separate the nodes with high residual from the nodes with low residual. Having criteria or not, then this procedure generate a full binary trees of nodes.

If a criteria is not used to select the bad nodes – and "good parents" –, then the binary tree will be perfect tree. In each borning step of the procedure, each leaf of each tree have to be analysed.

The above iterative algorithm is good, if the generated, growing multigrid RBF system converge to the $f(x)$ function. In order to have such an algorithm, a good *selector* and a good *borning* procedure must be defined.

3.2 Definition of the selector procedure

The selector procedure have to take different between nodes with good and bad approximating ability. The quality of approximation can estimated with the w_i^j nodal weights, defined in the previous section. In our examples, we used a very simple criteria to choose leaf to born.

Definition 2 *Let C^j the set of nodal centers at the j^{th} step of the iteration. Let us define L^j set*

$$L^j := \left\{(c_k^l, l = 1..j, k = 1..N_l; c_k^l \text{ is leaf of the binary tree})\right\}$$

Definition 3 *The w_i^j nodal weights are assumed to known and L^j is the set of leaf at step j^{th}. The sum and avaraged error of j^{th} step is*

$$\begin{aligned} SR^j &= \sum_{c_k^l \in L^j} w_k^l \\ AR^j &= \frac{SR^j}{num\left(L^j\right)} \end{aligned}$$

The sum of weight is a good measure of the quality of the itterative procedure. If the procedure converge to the $f(x)$ function then $\lim_{j\to\infty} SR^j = 0$. In the other hand the avaraged weight is a good criteria to choose the between good and bad nodes. In our example we decided by the next criteria:

Criterion 4 *Let $c_k^l \in L^j$ – it is a leaf. If $w_k^l \leq AR^j$ then the quality of c_k^l node is good, else it is bad and it must born two child.*

3.3 Definition of borning procedure

We assume that at the j^{th} step of the itterative algorithm the j^{th} approximated solution, the w_i^j nodal weights are known and c_i^j is a center with bad quality node. To born two new nodal children we define some properties of the center.

Remember, we defined the X_i^j subset of the sampling points around the c_i^j center point:

$$X_i^j := \left\{ \mathbf{x}_k \in \Omega, \varphi_i^j \left(\mathbf{x}_k \right) \neq 0 \right\}$$

It can be seen, in the case of RBF with compact support $X_i^j \subset X$, else $X_i^j \equiv X$. Let us assume that the serie of $\left\{C^1\right\}, .., \left\{C^j\right\}$ sets of centers, $\left\{d^1\right\}, .., \left\{d^j\right\}$ are known. Our goal is to determine $\left\{C^{j+1}\right\}$ and $\left\{d^{j+1}\right\}$ in order to have less residual then the last level.

Definition 5 *Let $\left\{C^j\right\}$ is the set of centers on level the j^{th}. Let X_i^j the set of sample points "near" to $\mathbf{c}_i^j$ center. Let the first order momentum of the X_i^j set is defined by*

$$\mathbf{S}_i^j := \sum_{\mathbf{x}_k \in X_i^j} \mathbf{x}_k \cdot \left\| R^j \left(\mathbf{x}_k \right) \right\|$$

Definition 6 *Let w_i^j and $\mathbf{S}_i$the weight and first order momentum of c_i^j, respectively. Let the centroid of the X_i^j set is defined by*

$$\mathbf{s}_i^j = \frac{\mathbf{S}_i^j}{w_i^j}$$

Definition 7 *Let $\mathbf{s}_i^j$ is the centroid of w_i^j residuals of c_i^j node. Let us define the inertial matrix of the X_i^j set:*

$$\mathbf{\Theta}_i^j := \sum_{\mathbf{x}_k \in X_i^j} \left(\mathbf{x}_k - \mathbf{s}_i^j \right) \times \left(\mathbf{x}_k - \mathbf{s}_i^j \right) \cdot \left\| w_i^j \right\|$$

Lemma 8 *The $\mathbf{\Theta}_i^j$ matrix is symmetric and positive definite.*

Lemma 9 *Every eigenvalue of the $\mathbf{\Theta}_i^j$ matrix are real.*

The calculation of eigenvalues and eigenvectors of $\mathbf{\Theta}_i^j$ matrix is trivial. Let we assume that, in three dimensional space the eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and $\mathbf{e}_1$, $\mathbf{e}_2$ és $\mathbf{e}_3$ eigenvectors are calculated. The plane with normal vector $\mathbf{e}_1$ and the point $\mathbf{s}_i^j$ is able to split the X_i^j subset of sampling points into two $X_{i,son}^j$ and $X_{i,daughter}^j$ sub-subsets.

Definition 10 *Let $\mathbf{e}_1$ the eigenvector of $\mathbf{\Theta}_i^j$ inertial matrix with the highest eigenvalue. Let , $\mathbf{x}_k$ a k^{th} sampling point pont and $\mathbf{s}_i^j$ is the centroid of X_i^j. Let us define the $X_{i,son}^j$ and $X_{i,daughter}^j$ subsets with the criteria*

$$\begin{aligned} X_{j,son}^i &: = \left\{ \mathbf{x}_k | \forall \mathbf{x}_k \in X_i^j, \left(\mathbf{x}_k - \mathbf{s}_i^j \right) \cdot \mathbf{e}_1 > 0 \right\} \\ X_{i,daughter}^j &: = \left\{ \mathbf{x}_k | \forall \mathbf{x}_k \in X_i^j, \left(\mathbf{r}_j - \mathbf{s}_i^j \right) \cdot \mathbf{e}_1 \leq 0 \right\} \end{aligned}$$

The last step is the determination of the nodal parameters of the new children. In the case of RBF with noncompact support, it seems to be a good choise to use the

$$a^i_{j,son} = a^i_{j,daughter} = 0.8 \cdot a^i_{j,son}$$

In the case of compact support a more carefull way must be used. If we do not use the appropriate value of new parameters, then some sample points can fall out from our horizon and their residual can never be eliminated. To avoid this problem we suggest the value $a^i_{j,son} = 1.2 \cdot r_{son,\max}$, where $r_{son,\max}$ is distance between the centroid of then "son" node and the farest sample point of $X^i_{j,son}$. Parameter of "daughter" can be calculated similarly.

4 Numerical examples

4.1 One dimensional function approximation test

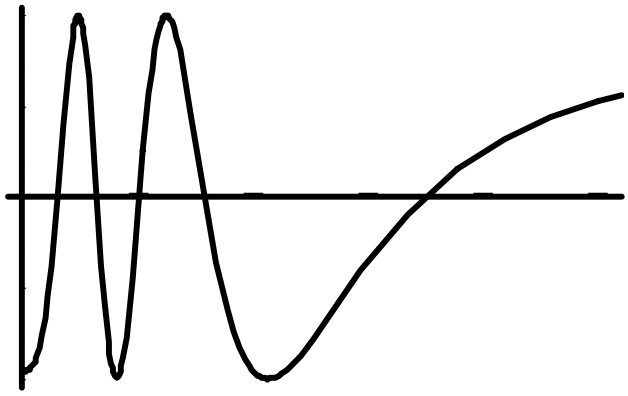
At first case, we tested our new method on some simple example. The approximated function was

$$f(x) := \sin\left(2 + \frac{15}{1+x^2}\right), x \in (0, 10)$$

We used *spline* type of RBF without any selection criteria during the iteration. To test the convergence the sum of the absolute value of residuals was used:

$$R := \sum_{i=1}^{NP} \left| f(x_i) - \widetilde{f}(x_i) \right|$$

The number of sample points was $NP := 2000$, uniformly distributed all over the domain of the problem. Next diagram shows the approximated function and the log-log diagram of R error as function of number of nodes.



First test: Approximated function

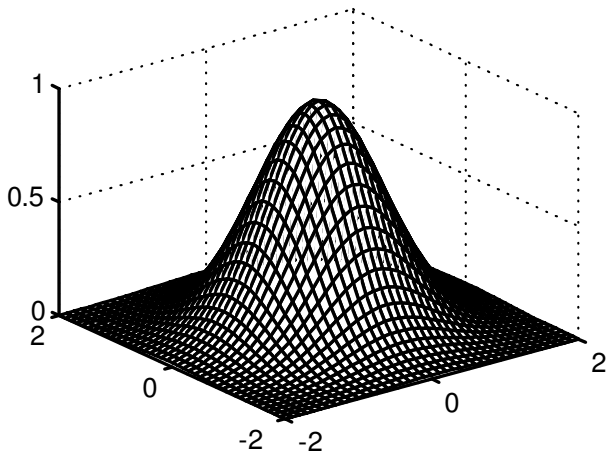
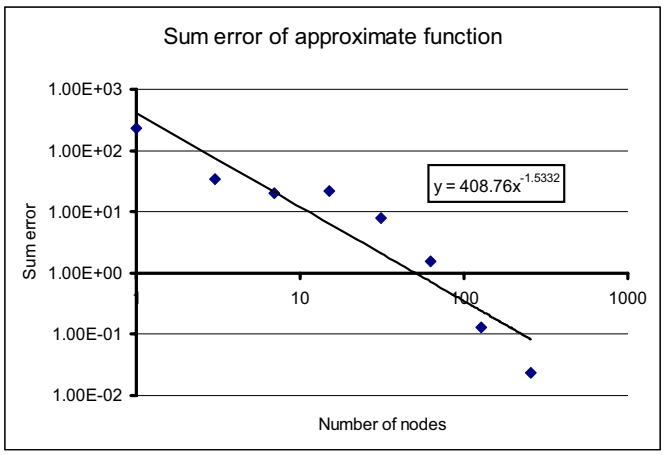


Figure 2: Two dimensional smooth function



Log-log diagram of absolute error as function of number of nodes

The one dimensional approximating itteration converged in hipebolic order.

4.2 Two dimensional test of smooth function

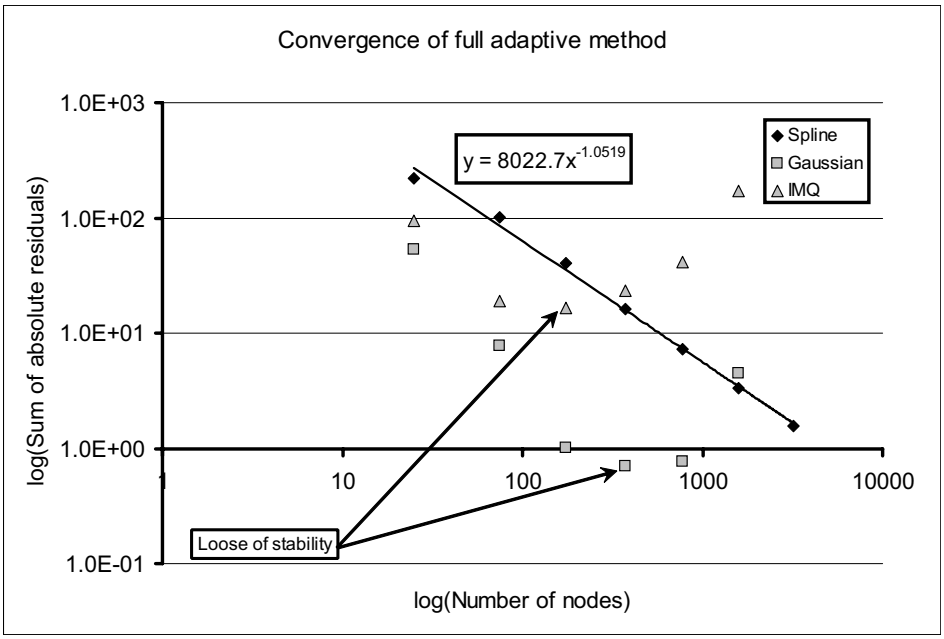
In our second example, the approximate procedure was tested on a smoth two dimensional function:

$$f(x,y) := e^{-\left(x^2+y^2\right)}, (x,y) \in ((-2,2) \times (-2,2))$$

It was tested with and without selection criteria on Gaussian, IMQ and spline type radial base functions. The set of sample points were position randomly and on a uniform 100×100 grid. The set of starting nodes were spread on a uniform grid of 5×5. Starting nodal parameter was $3 \times r_{\min}$ where $r_{\min}$ was the minimal distance between nodes.

Using Gaussian and IMQ base functions we had to recognize, with they the accuracy of the approximation is very good in the case of low number of nodes.

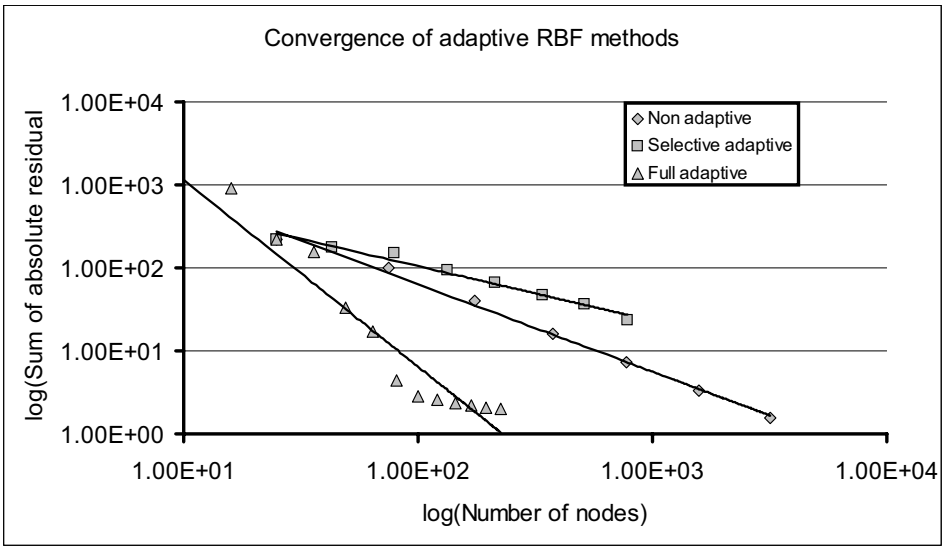
In the other hand during the refinement the accuracy did not become better and after some step the solution of system of equation became unstable. The instability is caused by that fact, in the case of this base functions the M matrix is full and condition number is bad.[IRODALOM!]



Convergence of approximation of 2D smooth function

Using of compact support spline base function the starting accuracy was not so fine, but the itteration converged very well and stayed stable. Additionaly, the M coeffcient matrix in this case is tipicaly sparse matrix and can be solved with special solvers. At the case of full matrixes usually the Gauss method is used to solve the system of equations. Cost of calculation in this case is $O\left(n^3\right)$, where n is the number of equations. If one has sparse matrix this cost can be decreased to $O\left(n \cdot \log n\right)$.

At second step the difference between full and selected method was tested. At this step we tested only the spline type radial basis function with compact support Setting were the same. The selection criteria of borning new nodes was the comparing the nodal residuals with avaraged residual.



Comapring the convergence of 2D smooth approximate

As we waited, and the literature had written, the full type of multigrid RBF converge faster.

4.3 Two dimensional test of function with discontinuity

In the third example a function, having first order discontinuity, was tested. In the analysis the convergency and the evolution of boring were observed. Our first example shows the convergency of

$$f\left(r,\alpha\right)=1+r^{2}\cdot\left(\frac{\alpha}{2\pi}-0.5\right)$$

function.

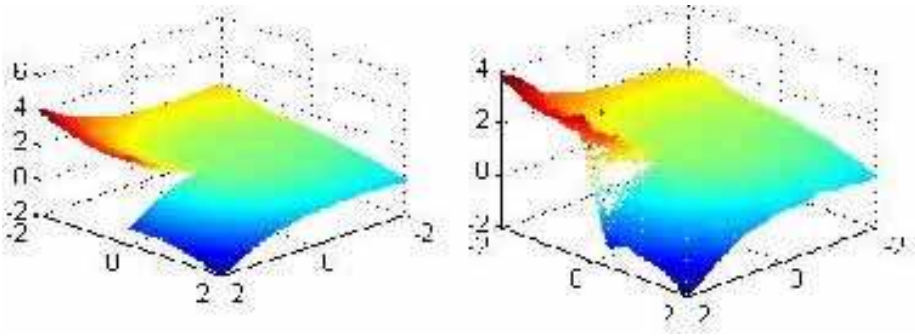


Figure 3: Approximating of function with 0th order discontinuity

The avarage of residual in this case converged exponencially.

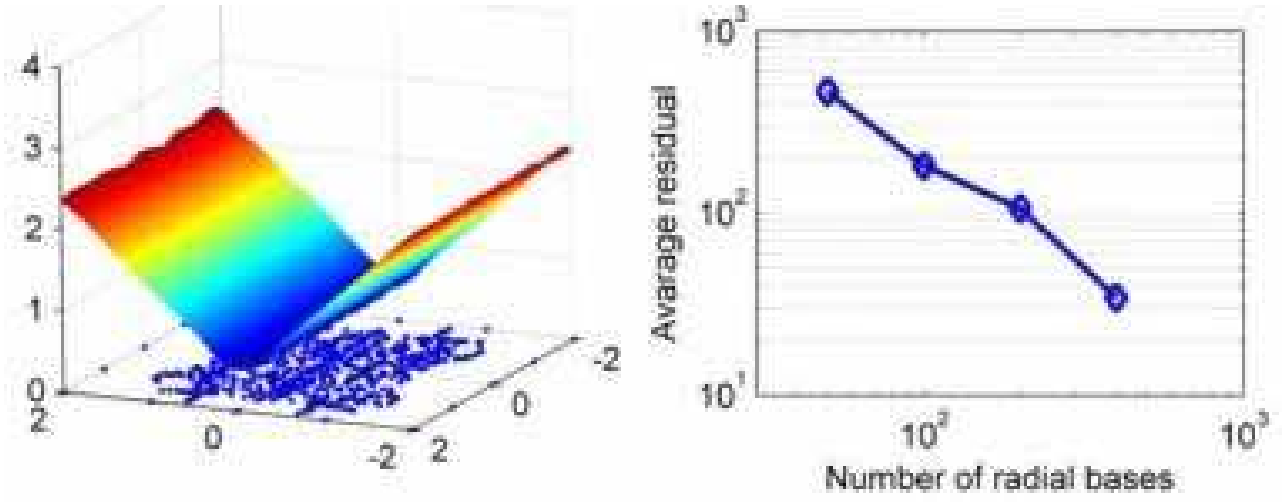
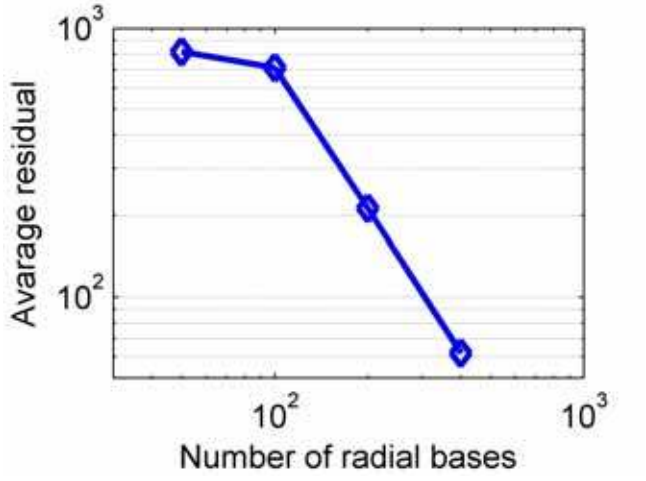


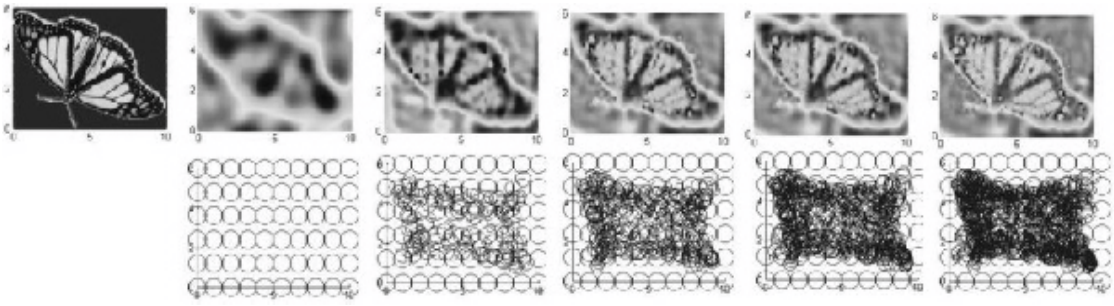
Figure 4: Approximation of ABS(x) function and avarage error



Convergence plot of avarage error

Testing of absolute value function showed similar convergence behaviour.

At last we show the evolution of the approximation - reconstruction - of a grayscale photo. During the iteration at each stage the actual state of the approximate picture was saved. After first rough steps, the smaller and smaller properties can be recognized on the pictures.



Reconstruction of "Monarch butterfly" photo

5 Conclusion

In this paper a new adaptive method was introduced, namely Inertial Adaptive Radial Basis Function method. After definition of multilevel RBF method, the definition of inertial properties of residual was introduced. The set of sampling points was divided into two subset with the first eigen vector of local inertial matrix was We split the radial basis support into parts. Growing of number of levels caused convergence of approximation. The convergence properties was tested on one and two dimensional continous and discontinous test functions. At last, a short series of picture showed the evolution of the approximation on a photo with small features.

Recently we test the new IARBF method on solving boundary value problems of differential equations. The IARBF method seems to have the ability to solve such problems on adaptive way, with respectable speed and stability.

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